Chapter 1

Introduction

Solving problems of wave propagation in turbulence is a field that occupies the services of a small group of researchers. The methods used in this community and the results obtained are not generally known by researchers in other communities. The main reason is that the field is considered difficult, and if there is not an obvious need to investigate the effects of turbulence, they are neglected. The difficulty arises from the need to solve stochastic differential equations. Advances made by Tatarski and Rytov reduce problems to multiple integrals. These integrals are often difficult to evaluate since fractional exponents of functions appear in integrands. The final step in most cases is to evaluate these integrals numerically and to present the results as parametric curves. Many cases are run to develop some insight into how a quantity of interest varies with parameters. Becoming an expert in this field requires a great deal of time to become familiar with these graphical results so that one has some insight into various effects.

As pointed out above, there is a formalism for reducing a problem to quadratures. This process is lengthy, and there are several ways of doing it. Different workers use different methods to get at the same result. This makes it difficult for the novice to understand the literature and to realize that there is some underlying order. This discourages a person with only a casual interest from developing a facility in this field. It was to make the solution of these problems more algorithmic that the methods expounded in this book were developed.

In this book I use the Rytov approximation to reduce a very general problem to a triple integral. I develop techniques that allow one to evaluate these integrals analytically.

The integrals that one encounters contain products of functions of which one or more is a Bessel function. Workers in the field look for these integrals in integral tables, and if unsuccessful, resort to numerical analysis. Even numerically, some of these integrals are difficult to evaluate. The integrand is often either the product of a function that goes to infinity multiplied by one that goes to zero at one of the integration limits, or the difference of two functions that each lead to a divergent integral. Great care must be exercised in evaluating these integrals.

The techniques developed in this book provide a recipe for obtaining analytical solutions. There is a saying, "You don't get something for nothing." Indeed, there is a price to be paid for being able to solve these problems more easily: The method uses mathematical techniques that are not generally taught to physicists and engineers, and one has to be willing to learn these techniques. One has to become familiar with Mellin transforms, gamma functions, generalized hypergeometric functions, asymptotic techniques, and pole-residue integration in one or more complex planes. Once these techniques are mastered, one can quickly solve complicated problems of wave propagation in turbulence. In addition, learning these techniques has some important ancillary benefits that are also discussed in this book. The techniques can be applied to most integrals in tables. In addition, one can easily evaluate integrals that are not in the tables. Since complicated integrals appear in many fields, the technique has wide applicability. Also Mellin transform techniques can be applied to solve differential and integral equations and have found many other applications as well.

An objection voiced against this method is that expressing an integral in terms of generalized hypergeometric functions is not very useful. The argument against the use of these functions is rapidly fading with the increasing accessibility of personal computers. These functions can be evaluated easily with recursion relations; in addition, they have become even easier to use since the introduction of computer algebra programs on personal computers that manipulate these functions just as easily as trigonometric functions. It is easy to evaluate expressions in which these functions appear, and there is no reason to eschew them any more than Bessel functions.

1.1 Book Plan

In this book I systematize the solution of many practical problems in wave propagation through turbulence. In Sec. 2 of this chapter the Mellin transform and the Mellin convolution integrals are introduced. A short table of Mellin transforms that are used to solve propagation problems in turbulence is also given.

In Chap. 2 the Rytov approximation for wave propagation is introduced as is the concept of the turbulent spatial spectrum. I describe a fairly general problem of wave propagation in turbulence that is solved with the Rytov approximation. This problem involves the difference between the log-intensity or phase of two waves that can be propagating in different directions or be displaced from each other and have different focal distances. I show that solutions to problems of this kind can be reduced to triple integrals; one integral along the propagation direction and two along the spatial transform wavenumbers of the wave. Most problems encountered in practice are special cases of this general problem.

For the Rytov approximation to be valid, the log-amplitude variance σ_χ^2 must be less than approximately 0.35. Above that value the scintillation is referred to as saturated, and it does not increase much with increasing propagation distances. The formulas using the Rytov approximation incorrectly result in scintillation values that continue to grow without bound. Even when the Rytov derived formulas give a log-amplitude variance result above the saturated value, the values obtained for the phase variance are generally valid, and the methods given in this book give correct answers.

For some problems, such as those that rely on the amplitude returns for tracking or in which one must determine the bit-error rate in an optical communications system, one needs the probability distribution function of the amplitude. For aperture diameters small compared to the coherence diameter, the main effect is amplitude fluctuations in the receive aperture since phase fluctuations are small. In this case it is found that the log-amplitude fluctuations are Gaussian and thus the intensity has a log-normal distribution that is specified by the single parameter of the scintillation index. For very large diameters the amplitude is the sum of a large number of terms that have a random phase. This leads to a negative exponential distribution of the intensity.

In the intermediate range of diameter the situation is more complicated. The tilt of the beam can produce displacements that are comparable or larger than the beam diameter. The scintillation and the intensity distribution function depend on the degree of tracking. The distribution function is characterized by several parameters. It has not been derived analytically and is typically specified by fits to data generated by computer simulations. This has led to semi-heuristic formulas for these parameters that are given in *Parenti et al.* (2006).

In addition, the first-order Rytov solution cannot be used in the saturated region. For these problems, one approach is to use a wave-optics code in which the wave is propagated forward in a computer simulation. Another approach is to use amplitude statistics that have been obtained experimentally or by computer simulations. *Nakagami* (1960) found that the amplitude distribution is gamma-gamma. The scintillation in the saturated region is surprisingly dependent on the outer scale size and is strongly affected by the inner scale size. The use of a semi-heuristic approach to determining this distribution is given in *Andrews* (2001) and *Tyson* (2002).

The second order statistics that are examined in this book are sufficient for finding the variances and power spectral densities of quantities of interest. One needs to find the fourth-order moments to determine the distribution functions. Methods of doing that and analyzing saturated scintillation are not discussed in this book.

Variations in the amplitude for coherent illumination are produced not only by scintillation, but also from speckle. This coherent interference effect can be important. It has been treated elsewhere and is not considered in this book.

In the unsaturated region, the expression for the variance of the phase or log-amplitude of a wave propagating from $z = 0$ to $z = L$ is of the form

$$\sigma^2 = 0.2073\, k_0^2 \int_0^L dz\, C_n^2(z) \int d\boldsymbol{\kappa}\, f(\kappa)\, g\left(\boldsymbol{\kappa}, z\right), \tag{1.1}$$

where $k_0 = 2\pi/\lambda$ with λ the propagation wavelength, $C_n^2(z)$ is the turbulence strength, $f(\kappa)$ is the normalized turbulence spectrum in transverse spatial transform space, and $g(\boldsymbol{\kappa}, z)$ is a filter function that depends on the particular problem. The κ integration is over the entire two-dimensional space.

Different problems simply necessitate the use of different filter functions. There is the simple interpretation that for a particular problem the original turbulence spectrum is filtered. One must then integrate over the residual spectrum and along the path to find the variance.

By making a change of the spatial spectrum variables in these integrals to two new variables, one of which is frequency, one can find the power spectral density for the particular problem.

There are characteristic categories of problems. The characteristic filter functions are derived in Chap. 3. For example, the filter functions for Zernike modes, such as piston and tilt on filled and annular apertures, are derived. I show that relative displacement of two waves (anisoplanatism) can be represented as a filter function. Filter functions corresponding to a finite size source or a finite size receiver are also derived. The difference between the phase of a focused and a collimated wave affects the performance of many interesting systems, and filter functions for this problem are also evaluated.

Many applications discussed in this book are for adaptive-optics systems. An adaptive-optics system tries to correct for the effects of turbulence in an object's image or in the projection of a laser beam, and its effect on the turbulence also can be represented by filter functions. A generic system is shown in Fig. 1.1.

As I have said, this formalism reduces problems to a triple integral. The evaluation of the double integral in wavenumber space is discussed in detail in this book and the method presented has applications in many fields. The evaluation of the one remaining integral over the propagation path can be done analytically for some turbulence distribution models and is performed numerically for others. There are no numerical difficulties with this last integral.

The evaluation of these integrals is considered in Chap. 4 for the simplest problems in which the solution is obtained by table lookup of Mellin transforms. These integrals are of the form

$$\int_0^\infty dx\, h(x)\, x^a, \tag{1.2}$$

where $h(x)$ is typically a trigonometric or Bessel function or the square of one, or it is an algebraic function. In general the integral can be performed analytically when the function is any Meijer G-function. These integrals lead to important results for the effect of turbulence on wave propagation; for example, tilt jitter or other Zernike modes, residual phase variance due to a finite-bandwidth adaptive-optics system, and scintillation.

In order to solve more difficult problems, integrals whose integrand contains the product of two functions must be evaluated. These integrals contain one free parameter and are evaluated by using the Mellin convolution theorem to convert the integral in real space into one in the complex plane. These integrals are of the form

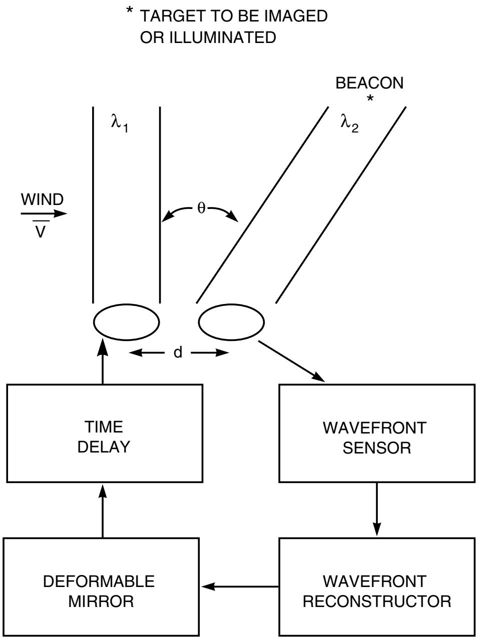

Figure 1.1. An adaptive-optics system. A beacon sends a beam through the atmosphere, and the turbulence impresses a phase aberration on it. This phase front is sampled by a wavefront sensor, which typically responds to local wavefront tilts in sections of the aperture referred to as subapertures. Phase is reconstructed from tilts in a wavefront reconstructor, and its conjugate is impressed on a deformable mirror. This mirror is in the path of the imaging or beam projection system, with the net result that turbulence phase aberrations are corrected.

$$\int_0^\infty dx\, h(x)\, g\left(x/y\right) x^a, \tag{1.3}$$

where y is the free parameter, and the functions are of the same type as in (1.2).

The method for evaluating this integral, which uses pole-residue integration, is developed in Chap. 5 for a positive-real parameter. The case of non-positive and complex parameters is developed in Chap. 9. Asymptotic solutions are developed. Some programs such as Mathematica from Wolfram Research, Inc. evaluate these integrals. The asymptotic results are not provided, and there may be difficulty evaluating some of the functions for large values of the parameter; these values are often required for the z integration. The evaluation of these integrals analytically is time consuming and error prone. When Mathematica can be used it is a a great aid, which enables solutions and power series representations to be found quickly.

In Chap. 6 I address a host of important problems in wave propagation through turbulence in which the integral is of the form in (1.3). The following results are obtained: It is shown that the finite outer scale of turbulence can have a significant effect on tilt jitter. The effect of inner scale is shown to limit the maximum tilt that can be measured on a small aperture. It is also shown that the inner-scale size can significantly reduce the scintillation. Tilt on an annulus is considered, and it is shown that tilt jitter does not change much from that of a filled aperture until the central obscuration becomes very large. The scintillation measured from a finite size source or by a finite size receiver is shown to be considerably reduced from that of a point source or point receiver. Expressions for characteristic sizes of the source or receiver that significantly reduce scintillation are developed. The power spectral density of turbulence caused by wind transport or receiver motion is calculated. I calculate how scintillation is affected by an adaptive-optics system. The correlation function of focus is evaluated. The residual phase is calculated when one corrects the turbulence on a collimated beam by using a focused source as a reference. The general solution for Zernike anisoplanatism is given. The residual piston and tilt for distributed and offset sources are calculated.

In addition to calculating wave variances, one can use these techniques to find the Strehl ratio as discussed in Chap. 7. Analytical expressions for the Strehl ratio cannot be calculated for every case of interest. For that reason, an approximation is derived for the Strehl ratio with anisoplanatism that is much better than the Maréchal approximation. This formula is applied to anisoplanatism caused by parallel displacement, angular displacement, time displacement, chromatic displacement, and the combination of several such effects. Beam shapes for these cases are calculated in Chap. 11.

For many problems the various approximations for the Strehl ratio are not valid. In addition, when tilt is removed the structure function is a function of position, and the simpler formula to calculate the Strehl ratio is not applicable. These problems must be solved by numerical techniques. Examples of doing that are given in Sec. 7.5.

Propagation of a Gaussian wave results in integrals that have complex or negative parameters. The theory to evaluate these integrals is developed in Chap. 8 and applied in Chap. 9 to the scintillation of a Gaussian beam.

For more complicated problems there can be two or more parameters in the integral. These integrals are of the form

$$\int_0^\infty dx\, h(x)\, g_1\left(x/y_1\right) \ldots g_N\left(x/y_N\right) x^a, \tag{1.4}$$

where the functions h and g are of the same type as in (1.3). These integrals can be converted into integrals in several complex planes with the Mellin convolution theorem. Pole-residue techniques apply to this class of problems. There are complications in trying to decide which poles in the multidimensional space one should include when evaluating residues. A procedure is given in Chap. 10 that makes the process straightforward. I also develop a technique to evaluate both the series and steepest-descent contributions to an asymptotic solution.

For these more complicated problems, the series solutions sometimes do not converge for all parameter ranges. The source of this difficulty is illustrated in Chap. 10 by an integral whose integrand is a product of exponentials. A series of examples is then considered in Chap. 11. Several sample integrals in two or more complex planes are evaluated, and the results are shown to agree with integral tables. I then turn to four problems in wave propagation through turbulence that cover all parameter regions: Tilt anisoplanatism with outer scale is evaluated, and the effect of outer scale is shown to be not as large as its effect on pure tilt. The method of solution shows clearly where this difference arises. Tilt with both inner and outer scale present is calculated. The solution is shown to agree with earlier work when either effect is eliminated. Next, the low-frequency regime of the tilt power spectrum is shown to be the most greatly affected by outer scale. Finally, the phase structure and correlation functions with non-zero inner and outer scales are calculated.

In Chap. 12 I consider the beam shape for several problems. Appendix A gives the Mellin transforms for familiar functions that were not encountered in turbulence problems. Some commonly used functions are expressed as generalized hypergeometric functions in Appendix B.

1.2 Introduction to Mellin Transforms

In this section I introduce the Mellin transform, and discuss a few of its properties. The Mellin transform $H(s)$ of the function $h(x)$ is defined as

$$h(x) \to H(s) \equiv \mathcal{M}\left[h\left(x\right)\right] \equiv \int_0^\infty \frac{dx}{x} h(x)\, x^s. \tag{1.5}$$

The inverse transform is

$$h(x) = \frac{1}{2\pi i} \int_C ds H(s)\, x^{-s}. \tag{1.6}$$

The integration path is a straight line extending from $\eta - i\infty$ to $\eta + i\infty$ and crossing the real axis at a value of s for which the integral in (1.5) converges. The allowed values of η typically lie in a finite range and are specified in the Mellin transform tables. The Mellin transform tables may be augmented with the following three properties that are derivable from the Mellin transform definition:

$$h(a\,x) \to a^{-s} H(s), \quad a > 0, \tag{1.7}$$

$$x^b h(x) \to H(s+b), \text{ and} \tag{1.8}$$

$$h\left(x^p\right) \to H(s/p)/\left|p\right|, \quad p \neq 0. \tag{1.9}$$

For the general case one finds

$$x^b h\left(ax^p\right) \to \frac{1}{|p|\, a^{(s+b)/p}} H[(s+b)/p], \quad p \neq 0.$$

These relations are repeatedly used throughout the book to extend Mellin transforms from the tables to new functions. For instance, the Mellin transform of a Gaussian function is found from that of the exponential given in eq. 1.47 by the use of eq. 1.9 with $p = 2$ to obtain

$$\mathcal{M}\left[\exp\left(-x^2\right)\right] = 0.5\, \Gamma[s/2],$$

and eq. 1.7 with $a = 1/b$ to obtain

$$\mathcal{M}\left\{\exp\left[-\left(x/b\right)^2\right]\right\} = 0.5\, b^s \Gamma[s/2]. \tag{1.10}$$

The Mellin transform is the Laplace transform in the logarithm of the variable x. This is why the Mellin transform is sometimes referred to as the *log-polar transform*. The transform of an object that is scaled in size by a factor b is related to the unscaled transform by

$$H(s) = \int_0^\infty \frac{dx}{x} h(bx)\, x^s = b^{-s} \int_0^\infty \frac{dy}{y}\, h(y)\, y^s. \tag{1.11}$$

This scaling factor has magnitude unity if s is confined to the imaginary axis. This is analogous to the way a time shift results in the multiplication of a Fourier transform by a complex scaling factor. This scaling property makes Mellin transforms convenient for the analysis of chaos and of acoustical, radar, and visual signals.

Gamma functions appear repeatedly in this work, and I highlight a few of their properties. The gamma function is defined as

$$\Gamma[s] = \int_0^\infty dx\, \exp(-x)\, x^{s-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{s+n} + \int_1^\infty dx\, \exp(-x)\, x^{s-1}. \tag{1.12}$$

The argument s can be complex. The last integral is entire and the only singularities of the gamma function are simple poles at the negative integers $-n$ with residue $\left(-1\right)^n/n!$. The reciprocal of the gamma function can be shown to be an entire function; therefore, the only singularities of the ratio of gamma functions come from the numerator. *This property is central to allowing one to easily evaluate integrals in the complex plane whose integrand is the ratio of gamma functions.*

As one recalls from the Cauchy residue theorem in complex variable theory, the value of an integral in the complex plane that contains only simple poles within the closed integration region is equal to $2\pi i$ times the sum of the residues at the enclosed poles. If $f(z)$ does not have any singularities, then

$$\oint dz\, \frac{f(z)}{z - z_0} = 2\pi i f(z_0), \tag{1.13}$$

when the integration path, which encircles the pole in a clockwise fashion, encloses the pole at $z = z_0$. The value of the integral is equal to zero when the pole is not enclosed in the integration path.

Plots of the gamma function and its reciprocal are shown in Fig. 1.2. The

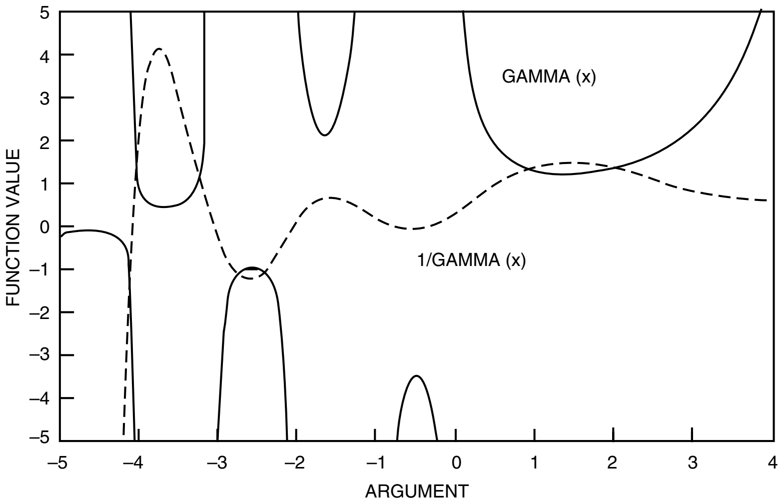

Figure 1.2. Gamma function and its reciprocal versus its argument.

recursion relation for gamma functions is

$$\Gamma[a+1] = a\Gamma[a]. \tag{1.14}$$

With the relation $\Gamma[1] = 1$, for integer a this gives $\Gamma[a+1] = a!$. With this relation the gamma function at noninteger values can be evaluated on some hand calculators.

If the argument of s in all the gamma functions is unity, then the integral can be expressed as a sum of generalized hypergeometric functions. This form is desirable because these functions can be plotted by standard computer programs. To change a gamma function argument with an integer multiplier to a gamma function with a unity factor, one uses the Gauss-Legendre multiplication formula

$$\Gamma\,[my] = m^{my-1/2}(2\pi)^{(1-m)/2}\prod_{k=0}^{m-1}\Gamma\,[y+k/m]. \tag{1.15}$$

The particular case of $m = 2$ is

$$\Gamma\,[2\,y] = 2^{2y-1}\Gamma\,[y]\,\Gamma\,[y+1/2]\,/\sqrt{\pi}. \tag{1.16}$$

To convert an argument with negative s to one with positive s, one uses the duplication formula

$$\Gamma\,[1-s]\;\Gamma\,[s] = \pi/\sin(\pi\,s). \tag{1.17}$$

For $s > 1$, the gamma function can be approximated by Stirling's formula

$$\Gamma[s] \sim \sqrt{2\pi}\, s^{s-1/2} \exp(-s) \left[1 + 1/(12\, s) + 1/(288\, s^2) + \cdots\right], \quad |\arg\{s\}| < \pi. \tag{1.18}$$

Table 1.1 is a list of Mellin transforms that are used to evaluate integrals encountered in wave propagation through turbulence. Additional Mellin transforms are given in Appendix A. These are only a small fraction of the 1200 transforms available in closed form. The notation of *Marichev* (1983) and *Slater* (1966) for the ratio of gamma functions used throughout the book is

$$\Gamma\left[\begin{matrix}\alpha_1, \ldots, \alpha_m\\ \beta_1, \ldots, \beta_n\end{matrix}\right] \equiv \frac{\Gamma[\alpha_1]\ \Gamma[\alpha_2] \ldots \Gamma[\alpha_m]}{\Gamma[\beta_1]\ \Gamma[\beta_2] \ldots \Gamma[\beta_n]}. \tag{1.19}$$

A new notation is introduced in eq. 1.50 in the Mellin transform table: An asterisk after a term in the gamma function signifies that the integration path passes between the first and second poles of that gamma function. The notation $\left(*^N\right)$ indicates that the integration path passes between the N^{th} and $N+1^{st}$ poles of the gamma function. No asterisk indicates that all poles are on one side of the integration path.

The table contains the Mellin transform of the Heaviside unit step function, defined by

$$\begin{aligned} U(x) &= 1 \ \text{ for } x > 0, \text{ and} \\ U(x) &= 0 \ \text{ for } x \leq 0. \end{aligned} \tag{1.20}$$

This function can be used to convert integrals with finite limits into ones with infinite limits that can be evaluated with the techniques developed in this book.

Mellin transforms are used in three ways to solve turbulence problems:

1. To evaluate integrals by table lookup.
2. To find a series representation for a function.
3. To evaluate complicated integrals.

The first application of Mellin transforms is simple yet important. Some integrals are easy to evaluate directly from the Mellin transforms. For instance, from eq. 1.48 one finds that

$$\int_0^\infty dx\, x^a \sin(x) = 2^a \sqrt{\pi}\, \Gamma\left[\begin{matrix}a/2 + 1\\ 1/2 - a/2\end{matrix}\right], \qquad \mathrm{Re}\{a\} < 0. \tag{1.21}$$

This technique is used to evaluate all the integrals in Chap. 4.

For some problems, i.e. those concerning anisoplanatic effects, one has to obtain the Mellin transform of the difference between a function and the first term of its power series. It is easy to show such a Mellin transform is that of the original function, except the integration path has moved over one pole. This is the analytic continuation of the integral. To illustrate, consider the transforms given in eq. 1.51 and eq. 1.57